

## AN APPROACH TO MEASURING THE RELATION BETWEEN RISK AND RETURN. BAYESIAN ANALYSIS FOR WIG DATA

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### ABSTRACT

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The main goal of this paper is an application of Bayesian inference in testing the relation between risk and return of the financial time series. On the basis of the Intertemporal CAPM model, proposed by Merton (1973), we built a general sampling model suitable in analysing such relationship. The most important feature of our model assumptions is that the possible skewness of conditional distribution of returns is used as an alternative source of relation between risk and return. Thus, pure statistical feature of the sampling model is equipped with economic interpretation. This general specification relates to GARCH-In-Mean model proposed by Osiewalski and Pipień (2000).

In order to make conditional distribution of financial returns skewed we considered a constructive approach based on the inverse probability integral transformation. In particular, we apply the hidden truncation mechanism, two approaches based on the inverse scale factors in the positive and the negative orthant, order statistics concept, Beta distribution transformation, Bernstein density transformation and the method recently proposed by Ferreira and Steel (2006).

Based on the daily excess returns of WIG index we checked the total impact of conditional skewness assumption on the relation between return and risk on the Warsaw Stock Market. Posterior inference about skewness mechanisms confirmed positive and decisively significant relationship between expected return and risk. The greatest data support, as measured by the posterior probability value, receives model with conditional skewness based on the Beta distribution transformation with two free parameters.

### KEY WORDS — SŁOWA KLUCZOWE:

bayesian model comparison, Bayes factors, GARCH models, skewness, fat tails  
wnioskowanie bayesowskie, czynnik Bayesa, GARCH, skośność, grube ogony

## 1. INTRODUCTION

The relationship between risk and return constitutes the foundation of financial economics. Numerous papers have investigated this trade-off testing the functional dependence of excess return on the level of risk, both measured by conditional expectation and conditional variance of aggregate wealth. According to Merton (1973), given risk aversion among investors, when investment opportunity set is constant, there is a positive relationship between expected excess return and the level of risk. Hence, it is possible to express the risk in terms of expected premium generated.

Historically, authors have found mixed empirical evidence concerning the relationship. In some cases a significant positive relationship can be found, in others it is insignificant and also some authors report it as being significantly negative. For instance, using monthly U.S. data French, Schwert and Stambaugh (1987) and also Campbell and Hentschel (1992) found a predominantly positive but insignificant relationship. Glosten, Jagannathan and Runkle found a negative and significant relationship on the basis of Asymmetric-GARCH model, instead of commonly used GARCH-in-Mean framework; see Engle, Lilien and Robins (1987). Scruggs (1998) summarises the empirical evidence of considered relationship.

Recent empirical work offers some resolutions to the conflicting results in the literature. Including into Merton (1973) Intertemporal Capital Asset Pricing Model (ICAPM) an additional risk factor, measured by long term government bond returns, Scruggs (1998) restores a positive relationship between risk and return. Pastor and Stambaugh (2001) considered a possibility of structural breaks in the risk-return relationship. Brandt and Kang (2004) investigated lead-lag correlations and find a strong negative dependence between the contemporaneous innovations and both the conditional mean and conditional variance of excess returns. Ghysels, Santa-Clara and Valkanov (2006) find a positive risk aversion coefficient using a mixed data sampling (MIDAS) approach to estimating conditional dispersion. Campbell and Hentschel (1992), Guo and Whitelaw (2003) and Kim, Morley and Nelson (2004) find a positive relationship when volatility feedbacks are incorporated. Also many authors investigated a nonlinear relationship between market risk and return, finding some evidence confirming positive risk premium; see Pagan and Hong (1990) for nonparametric approach and for other methods: Backus and Gregory (1993), Whitelaw (2000) and Linton and Perron (2003).

The main goal of this paper is an application of Bayesian model comparison, based on the posterior probabilities, in testing the relation between risk and excess return of the financial time series. We revisited ICAPM model in order to investigate the empirical importance of the skewness assumption of the conditional distribution of excess returns. On the basis of the ICAPM model, we built a general sampling model suitable in estimating risk premium. The most

important feature of our model assumptions is that the possible skewness of conditional distribution of returns is used as an alternative source of relation between risk and return. Thus, pure statistical feature of the sampling model is equipped with economic interpretation. This general specification relates to GARCH-In-Mean model proposed by Osiewalski and Pipień (2000). In order to make conditional distribution of financial returns skewed we considered a constructive approach based on the inverse probability integral transformation. In particular, we apply the hidden truncation mechanism, two approaches based on the inverse scale factors in the positive and the negative orthant, order statistics concept, Beta distribution transformation, Bernstein density transformation and the method recently proposed by Ferreira and Steel (2006).

Based on the daily excess returns of index of the Warsaw Stock Exchange we checked the total impact of conditional skewness assumption on the relation between return and risk on the Warsaw Stock Market. On the basis of the posterior probabilities and posterior odds ratios, we test formally the explanatory power of competing, conditionally fat tailed and asymmetric GARCH processes. Additionally we present formal Bayesian inference about conditional asymmetry in all competing specifications on the basis of the skewness measure defined by Arnold and Groenveld (1995).

## 2. CREATING ASYMMETRIC DISTRIBUTIONS

The unified representation of univariate skewed distributions that we study is based on the inverse probability integral transformation.

The family  $IP = \{\varepsilon, \varepsilon_s; \Omega \rightarrow R\}$ , with the representative density  $s(.|\theta, \eta_p)$  is called the skewed version of the symmetric family  $I$  (of random variables with unimodal symmetric density  $f(.|\theta)$  and distribution function  $F$ , such that the only one modal value is localised at  $x = 0$ ) if  $s$  is given by the form:

$$s(x|\theta, \eta_p) = f(x|\theta) \cdot p(F(x|\theta)|\eta_p), \text{ for } x \in R. \quad (1)$$

As seen from (1) the asymmetric distribution  $s(.|\theta, \eta_p)$  is obtained from  $f(.|\theta)$  by applying the density  $p(.|\eta_p)$  as a weighting function. The resulting family  $IP$  is parameterised by two vectors,  $\theta$  and  $\eta_p$ , where first is strictly inherited from the symmetric family  $I$ , while  $\eta_p$  contains specific information about the skewing mechanism. The most important feature of (1) is that the distributions  $s$  and  $f$  are identical if and only if  $p(.|\eta_p)$  is the density of the uniform distribution over the unit interval; i.e.  $s = f$  iff  $p(y|\eta_p) = 1$ , for each  $y \in (0, 1)$ .

Within the general form (1) several classes of distributions  $P$  have been imposed on some specific families of symmetric random variables. The first approach of making distribution  $F(.|\theta)$  skewed applied hidden truncation ideas.

The skew-Normal distribution proposed by Azzalini (1985) constitutes the first explicit formulation of such a skewing mechanism. In general this approach assumes, that:

$$s(x|\theta, \gamma_2) = 2 \cdot f(x|\theta) \cdot F(\gamma_2 \cdot x|\theta), \text{ for } x \in R, \quad (2)$$

where  $\gamma_2 \in R$  is the only one parameter which governs the skewing mechanism;  $\eta_p = (\gamma_2)$ . In this case  $p(y|\gamma_2) = 2 \cdot F(\gamma_2 \cdot F^{-1}(y)|\theta)$ , for  $y \in (0, 1)$ , and hence in (2) positive and negative values of  $\gamma_2$  define right and left skewed distributions. Since, for each  $y \in (0, 1)$ , it is true that  $p(y|0) = 2 \cdot F(0 \cdot F^{-1}(y)|\gamma_2 = 0) = 1$ , the case  $\gamma_2 = 0$  leads to symmetry in (2).

As an alternative Jones (2004) proposed to apply the family of Beta distributions in order to define  $p(.|\eta_p)$ . This is a formal application of the distribution of order statistics in skewing the family of random variables  $I$ . In particular  $s(x|\theta, \gamma_3)$  can be defined as follows:

$$s(x|\theta, \gamma_3) = f(x|\theta) \cdot Be(F(x|\theta) | \gamma_3, \gamma_3^{-1}), \text{ for } x \in R, \quad (3)$$

where  $Be(y|a, b)$  is the value of the density function of the Beta distribution with parameters  $a > 0$  and  $b > 0$ , calculated at  $y \in (0, 1)$ . Since  $Be(.|1, 1)$  defines the density of the uniform distribution, we obtain, that for  $\gamma_3 = \gamma_3^{-1} = 1$  the density  $s$  is symmetric. In (3) there is still only one parameter  $\gamma_3 > 0$ , which defines the type of asymmetry. If  $\gamma_3 > 1$ , then  $s$  is right asymmetric, while  $\gamma_3 < 1$  constitutes skewness to the left.

The family IP of skewed distributions proposed in (3) can be generalised, by incorporating Beta distribution with two free parameters  $a > 0$  and  $b > 0$ . This leads to the following form of  $s$ :

$$s(x|\theta, \eta_p) = f(x|\theta) \cdot Be(F(x|\theta) | a, b), \text{ for } x \in R. \quad (4)$$

In this case the vector  $\eta_p = (a, b)$ , which governs skewness, contains two parameters. If  $a = b = 1$  we retrieve symmetry, while  $a < b$  or  $a > b$  defines left or right skewness. It can be shown that the skewing mechanism (4) in case, when  $I$  is the family of Student- $t$  distributions yields skewed Student- $t$  family proposed by Jones and Faddy (2003).

Another method for introducing skewness into an unimodal distribution is based on the inverse scale factors on the left and on the right side of the mode of the density  $f(.|\theta)$ . Investigating this concept Fernández and Steel (1998) proposed skewed Student- $t$  family of distributions with the density  $f_{skv}(.|\nu, 0, 1, \gamma_1)$  defined as follows:

$$f_{skv}(x|0, 1, \nu, \gamma_1) = \frac{2}{\gamma_1 + \gamma_1^{-1}} \{ f_t(x \cdot \gamma_1 | 0, 1, \nu) \cdot I_{(-\infty, 0)} + f_t(x \cdot \gamma_1^{-1} | 0, 1, \nu) \cdot I_{(0, +\infty)} \},$$

where  $f_i(x|\mu, h, \nu)$  denotes the value of the density function of the Student- $t$  distribution with  $\nu > 0$  degrees of freedom, modal parameter  $\mu \in R$  and inverse precision  $h > 0$ , calculated at  $x \in R$ . The approach studied by Fernández and Steel (1998) can be applied to any family  $I$  of symmetric distributions by defining in (1) the following skewing mechanism for each  $y \in (0, 1)$ :

$$p(y|\gamma_1) = \frac{2}{\gamma_1 + \gamma_1^{-1}} \frac{\{f(\gamma_1 \cdot F^{-1}(y))I_{(0,0.5)} + F(\gamma_1^{-1} \cdot F^{-1}(y))I_{(0.5,1)}\}}{f(F^{-1}(y))}, \quad (5)$$

for  $\gamma_1 > 0$ . The resulting density  $s(\cdot|\theta, \gamma_1)$  is symmetric if  $\gamma_1 = 1$ , while  $\gamma_1 > 1$  or  $\gamma_1 < 1$  make distribution right or left skewed.

In the next approach we apply Bernstein densities, which are convex discrete mixtures of appropriate densities of Beta distribution. For posterior inference of such a family of distributions see Petrone and Wasserman (2002). The following form of  $p$  constitutes flexible skewing mechanism:

$$p(y|w_1, \dots, w_m) = \sum_{j=1}^m w_j Be(y|j, m-j+1), \quad y \in (0, 1),$$

where  $m > 0$ ,  $w_j > 0$ ,  $w_1 + \dots + w_m = 1$ .

The resulting  $s(\cdot|\theta, \eta_p)$  takes the form:

$$s(x|\theta, \eta_p) = f(x|\theta) \cdot \sum_{j=1}^m w_j Be(F(x|\theta) | j, m-j+1), \quad \text{for } x \in R, \quad (6)$$

where  $\eta_p = (w_1, \dots, w_{m-1})$ ,  $w_j \in (0, 1)$  for  $j = 1, \dots, m-1$ , and in (6)  $w_m = 1 - w_1 - \dots - w_{m-1}$ . Equal weights  $w_j = m^{-1}$  lead to the symmetry in (6). However, in general the resulting skewed densities (6) are often multimodal, especially for large values of  $m$ . The main disadvantage of skewing mechanism based on (6) is, that we loose some regularities of constructed family IP in favor of total flexibility in data fit.

Ferreira and Steel (2006) considered a constructive representation of the univariate skewness by defining a mechanism which does not depend on  $F$ , does not change location of the modal value and also keep the moment structure unchanged. Starting from a general form of skewing mechanism:

$$p(y|\gamma_4) = 1 + l(\gamma_4)[g(y|\gamma_4) - 1], \quad \gamma_4 \in R, \quad (7)$$

they proposed an approach of deriving appropriate  $l(\cdot)$  and  $g(\cdot|\gamma_4)$  meeting required properties. In particular, according to Ferreira and Steel (2006), it is possible to consider:

$$I(\gamma_4) = \frac{\left[ \frac{2}{\pi} \arctan(2\gamma_4/5) \right]^3}{1 - 2 \int_0^{0.5} g(t|\gamma_4) dt}$$

where  $g(y|\gamma_4) = h^*(y|\gamma_4)I_{[0, 0.5]} + [2 - h^*(y - 0.5|\gamma_4)I_{[0.5, 1]}]$ , for  $h^*$  of the following form:

$$h^*(y|\gamma_4) = h \left( \frac{\exp(\gamma_4 y - 1)}{2(\exp(0.5\gamma_4) - 1)} \right),$$

with  $h$  defined as a polynomial of order 2;  $h(z) = -32z^3 + 24z^2$ . Such a particular solution defines skewing mechanism equipped with properties restricted to the postulates of the construct. Ferreira and Steel (2006) do not present exhaustive characterisation of skewing mechanism (7), making their individual proposition focused only on modelling distributional skewness around the mode.

The next skewing mechanism is another example of an application of the inverse scale factors idea. Four years before Fernández and Steel (1998) published their skewed version of the  $t$  distribution, Hansen (1994) proposed the following simple generalisation of the Student- $t$  density (normalized to have unit variance):

$$s(x|0, 1, \nu, \gamma_5) = \begin{cases} bc \left( 1 + \frac{1}{\nu - 2} \left( \frac{bx + a}{1 - \gamma_5} \right)^2 \right)^{\frac{\nu+1}{2}} & \text{if } x < -\frac{a}{b} \\ bc \left( 1 + \frac{1}{\nu - 2} \left( \frac{bx + a}{1 + \gamma_5} \right)^2 \right)^{\frac{\nu+1}{2}} & \text{if } x \geq -\frac{a}{b}, \end{cases} \tag{8}$$

where  $\nu > 2$ ,  $\gamma_5 \in (-1; 1)$ ,  $a = 4 \cdot \gamma_5 \cdot c(\nu - 2)/(\nu - 1)$ ,  $b^2 = 1 + 3 \cdot \gamma_5^2 - a^2$  and

$$c = \frac{\Gamma(0.5(\nu + 1))}{\Gamma(0.5\nu) \sqrt{\pi(\nu - 2)}}.$$

If  $\gamma_5 = 0$ , then the function  $s(x|0, 1, \nu, 0)$  represents the density of the symmetric Student- $t$  distribution with  $\nu > 2$  degrees of freedom, zero mode and unit variance, while  $\gamma_5 \in (-1, 0)$  and  $\gamma_5 \in (0, 1)$  enable for left and right asymmetry respectively. Here we will show, that the Hansen skewed Student- $t$  density can be treated within (1) as a result of imposing a particular skewing

mechanism based on the inverse scale factors on the left and on the right side of the mode. Let consider a random variable  $Z_s$  with the density of the following form:

$$f_s(x|0, 1, \nu, \gamma_s) = \{ f_{st}(x/(1-\gamma_s)|0, 1, \nu) \cdot I_{(-\infty, 0)}(x) + f_{st}(x/(1 + \gamma_s)|0, 1, \nu) \cdot I_{(0, +\infty)}(x) \}, \\ \gamma_s \in (-1; 1), \nu > 2,$$

where  $f_{st}(\cdot|0, 1, \nu)$  is the density of the Student- $t$  distribution with  $\nu > 2$  degrees of freedom, zero mean and unit variance. The mean  $E$  and the variance  $V$  of  $Z_s$  are given as follows:

$$E(Z_s) = a = 4 \gamma_s c(\nu-2)/(\nu-1)$$

$$V(Z_s) = b^2 = 1 + 3 \gamma_s^2 - a^2,$$

where:

$$c = \frac{\Gamma(0.5(\nu + 1))}{\Gamma(0.5) \sqrt{\pi(\nu - 1)}}.$$

It can be shown that the skewed density proposed by Hansen (1994) is the density of the random variable  $X_s$  obtained by standardisation of  $Z_s$ :

$$X_s = \frac{Z_s - a}{b}.$$

Consequently Hansen (1994) idea can be adapted to any symmetric and unimodal density  $f$  (with distribution function  $F$ ) by imposing the following skewing mechanism:

$$p(y|\gamma_s) = \frac{f\left(\frac{F^{-1}(y)}{1-\gamma_s}\right) I_{(0;0.5)}(y) + f\left(\frac{F^{-1}(y)}{1+\gamma_s}\right) I_{(0.5;1)}(y)}{f(F^{-1}(y))}, \text{ for } \gamma_s \in (-1; 1). \quad (9)$$

In spite of a very similar form of mechanism defining Fernández and Steel (1998) skewed Student- $t$  to the one related to Hansen (1994), both generalisations lead to different classes of asymmetric distributions with Student-type tails. But some equivalences can be discovered. In particular, for each  $\gamma_s \in (-1, 1)$ , there exist only one  $\gamma_2^* > 0$  (namely  $\gamma_2^* = \{(1 + \gamma_s)/(1 - \gamma_s)\}^{0.5}$ ) such, that mechanisms  $p(\cdot|\gamma_s)$  in (9) and  $p(\cdot|\gamma_2^*)$  in (5) generate skewed densities with the same ratio of the probability mass on the left and on the right side of the mode.

In the next section we present basic model framework, which is a starting point in generating conditionally heteroscedastic models for daily returns. In order to create the set of competing specifications, we make use of all presented skewing mechanisms.

### 3. BASIC MODEL FRAMEWORK AND COMPETING SKEWED CONDITIONAL DISTRIBUTIONS

Let denote by  $x_j$  the value of a stock or a market index at time  $j$ . The excess return on  $x_j$ , denoted by  $y_j$ , is defined as the difference between the logarithmic daily return on  $x_j$ , in percentage points ( $r_j = 100 \ln[x_j/x_{j-1}]$ ) and the risk free interest rate (denoted by  $r_f^j$ ), namely  $y_j = r_j - r_f^j$ . The voluminous literature focused on examination the relationship between risk and return bases on the Intertemporal Capital Asset Pricing Model, proposed by Metron (1973). According to the assumptions of Merton (1973) theory, expected excess return  $E$  is proportional to the standard deviaton  $D$  (both conditional with respect to the information set at time  $j$ , denoted by  $\psi_{j-1}$ ):

$$E(y_j|\psi_{j-1}) = \alpha^* D(y_j|\psi_{j-1}). \quad (10)$$

The coefficient  $\alpha^* > 0$  in (10) measures the relative risk aversion of the representative agent. Under assumption of the informational efficiency of the market, the information set at time  $j$  can be reduced to the history of the process of the excess return, namely  $\psi_{j-1} = (\dots, y_{j-2}, y_{j-1})$ . Consequently an econometric model of the relationship between risk and return should explain the properties of the conditional (with respect to the past of the process of  $y_j$ ) distribution of the excess return  $y_j$ . It is also of particular interest to find any linkage between expected excess return and the measure of dispersion of the distribution of  $y_j$  conditional to  $\psi_{j-1}$ . Following Engle, Lilien and Robins (1987), French, Schwert and Stambaugh (1987) and Osiewalski and Pipień (2000) we consider for  $y_j$  a simple GARCH-In-Mean process, defined as follows:

$$y_j = [\alpha + E(z_j)] h_j^{0.5} + u_j, \quad j = 1, 2, \quad (11)$$

where  $u_j = [z_j - E(z_j)] h_j^{0.5}$ , and  $z_j$  are independently and identically distributed random variables with  $E(z_j) < +\infty$ . The scaling factor  $h_j$  is given by the GARCH(1, 1) equation; see Bollerslev (1986):

$$h_j = \alpha_0 + \alpha_1 u_{j-1}^2 + \beta_1 h_{j-1}.$$

The specific form of the conditional distribution of  $y_j$  is strictly dependent on the type of the distribution of  $z_j$ . Initially, in model denoted by  $M_0$  we

assumed for  $z_j$  the Student- $t$  density with unknown degrees of freedom  $\nu > 1$ , zero mode and unit inverse precision:

$$z_j | M_0 \sim \text{St}(\nu, 0, 1), \nu > 1.$$

The density of the distribution of  $z_j$  is given as follows:

$$p(z_j | M_0) = f_t(z_j | 0, 1, \nu) = \frac{\Gamma(0.5(\nu + 1))}{\Gamma(0.5\nu) \sqrt{\pi\nu}} \left[ 1 + \frac{z_j^2}{\nu} \right]^{-(\nu+1)/2}. \quad (12)$$

Given model  $M_0$ ,  $E(z_j) = 0$ ,  $u_j = z_j h_j^{0.5}$ , and hence (11) reduces to the simpler form  $y_j = \alpha h_j^{0.5} + u_j$ . Let denote by  $\theta = (\alpha, \alpha_0, \alpha_1, \beta_1, \nu)$  the vector of all parameters in model  $M_0$ . Here the conditional distribution of the error term  $u_j$  is the Student- $t$  distribution with  $\nu > 1$  degrees of freedom, zero mode and inverse precision  $h_j$ :

$$p(u_j | \psi_{j-1}, \theta, M_0) = h_j^{-0.5} \cdot f_t(h_j^{-0.5} u_j | 0, 1, \nu), j = 1, 2, \dots$$

Consequently the following density represents conditional distribution of the excess return at time  $j$ :

$$p(y_j | \psi_{j-1}, \theta, M_0) = h_j^{-0.5} \cdot f_t(h_j^{-0.5} \cdot (y_j - \alpha h_j^{0.5}) | 0, 1, \nu), j = 1, 2, \dots$$

Given model  $M_0$  the expected excess return (conditional to the whole past  $\psi_{j-1}$ ) is proportional to the square root of the inverse precision  $h_j$ :

$$E(y_j | \psi_{j-1}, \theta, M_0) = \alpha h_j^{0.5}. \quad (13)$$

The parameter  $\alpha \in R$  captures the dependence between expected excess return and the level of risk both measured by  $E(y_j | \psi_{j-1}, M_0, \theta)$  and the scale parameter  $h_j^{0.5}$  respectively. Initially the relationship between risk and return stated in (10) relates to the conditional standard deviation as a measure of risk. In our approach, the relative risk aversion coefficient  $\alpha^*$  can be obtained by reparameterisation of the model in terms of variance, instead of introducing inverse precision. However our approach, based on the more general scale measure, enables to test Merton (1973) theory in case of more volatile excess returns, which do not possess conditional second moment.

Now we want to construct a set of competing GARCH specifications  $\{M_i, i = 1, \dots, k\}$  by introducing asymmetry into density of the conditional distribution of excess return,  $p(y_j | \psi_{j-1}, \theta, M_0)$ . The resulting asymmetric distributions are obtained by skewing the distribution of the random variable  $z_j$  according to

methods presented in the previous section. The resulting asymmetric density of  $z_i$  is of the general form related to the formula (1):

$$p(z | M_i) = f_i(z|0, 1, \nu) \cdot p [F_i(z) | \eta_i, M_i], \text{ for } z \in R, i = 1, 2, \dots, k,$$

where  $p (.|\eta_i, M_i)$  defines the skewing mechanism parameterised by the vector  $\eta_i$ , and  $F_i(.)$  is the distribution function of the Student- $t$  random variable with  $\nu > 1$  degrees of freedom parameter, zero mode and unit inverse precision. Consequently, the conditional distribution of the error term  $u_j$  in model  $M_i$  takes the form:

$$p(u_j | \psi_{j-1}, \theta, \eta_i, M_i) = h_j^{-0.5} \cdot f_i(h_j^{-0.5}u_j|0, 1, \nu) \cdot p[F_i(h_j^{-0.5}u_j) | \eta_i, M_i], j = 1, 2, \dots$$

This leads to the general form of the conditional distribution of daily excess return  $y_j$  in model  $M_i$ :

$$p(y_j | \psi_{j-1}, \theta, \eta_i, M_i) = h_j^{-0.5} \cdot f_i(h_j^{-0.5}(y_j - \mu_j)|0, 1, \nu) \cdot p[F_i(h_j^{-0.5}(y_j - \mu_j)) | \eta_i, M_i], \quad (14)$$

$$j = 1, 2, \dots,$$

where  $\mu_j = [\alpha + E(z_j)]h_j^{0.5}$ . As the first specification, namely  $M_1$ , we consider GARCH model with skewed Student- $t$  distribution obtained by the method investigated by Fernández and Steel (1998). The skewing mechanism  $p [.\eta_1, M_1]$  is given by the formula (5), where  $\eta_1 = \gamma_1 > 0$ , and  $\gamma_1 = 1$  defines symmetry (i.e.  $M_1$  reduces to the model  $M_0$  under restriction  $\gamma_1 = 1$ ). The model  $M_2$  is the result of skewing conditional distribution  $p (y_j | \psi_{j-1}, \theta, M_0)$  according to the hidden truncation method. In this case  $p [.\eta_2, M_2]$  is defined by (2),  $\eta_2 = \gamma_2 \in R$ , while  $\gamma_2 = 0$  defines symmetric Student- $t$  conditional distribution for  $y_j$ . In model  $M_3$  we apply Beta skewing mechanism with one asymmetry parameter. Density  $p [.\eta_3, M_3]$  is defined by (3), where  $\eta_3 = \gamma_3 > 0$ , and  $\gamma_3 = 1$  reduces our model to the case of  $M_0$ . Specification  $M_4$  is based on the Skewed Student- $t$  distribution proposed by Jones and Faddy (2003). In this case  $p [.\eta_4, M_4]$  is defined by the formula (4),  $\eta_4 = (a, b)$ , for  $a > 0$  and  $b > 0$  and  $a = b = 1$  reduces  $M_4$  to  $M_0$ . In model  $M_5$  we apply Bernstein density based skewing mechanism for  $m = 2$  parameters. It means that the skewing mechanism  $p [.\eta_5, M_5]$  is defined by the formula (6),  $\eta_5 = (w_1, w_2)$  and  $w_1 = w_2 = 1/3$  retrieves symmetry of the conditional distribution of  $y_j$ . In model  $M_6$  we applied a construct defined by Ferreira and Steel. Skewing mechanism  $p [.\eta_6, M_6]$  is defined by (7), while  $\eta_6 = \gamma_4 \in R$  and  $\gamma_4 = 0$  defines conditional symmetry. In specification  $M_7$  we considered Hansen (1994) skewed Student- $t$  conditional distribution, by applying mechanism  $p [.\eta_7, M_7]$  defined in (9). Here  $\eta_7 = \gamma_5 \in (-1, 1)$ , while  $\gamma_5 = 0$  reduces  $M_7$  to the case of conditional symmetry.

All formulated specifications assume, that the conditional distribution of  $y_j$  is heteroscedastic, where time varying dispersion measure  $h_j$ , defined by

GARCH(1, 1) specification, is a function of the whole past of the process. The degrees of freedom parameter  $\nu > 1$  enable for fat tails of  $p(y_j | \psi_{j-1}, \theta, \eta_i, M_i)$ . In each specification it is also possible to test whether the dataset supports conditional distribution with Gaussian-type tails (for  $\nu \rightarrow \downarrow$ ). The possible asymmetry of conditional distribution can be captured in all models by the presence of skewing mechanism. Additionally, skewness of the distribution of  $z_j$  in  $M_i$  generates nonzero expectation  $E(z_j) < + \downarrow$ . Consequently in (11):

$$E(y_j | \psi_{j-1}, \theta, \eta_i, M_i) = [\alpha + E(z_j)]h_j^{0.5}, \text{ for } E(z_j) \neq 0. \quad (15)$$

And hence for each specification  $M_i$ ,  $i = 1, 2, 3, 4, 5, 6, 7$ , conditional skewness of excess returns  $y_j$  can be interpreted as an additional source of the relationship between risk and return. This idea fully corresponds to Harvey and Siddique (2000), who emphasize, that systematic skewness is economically important and governs risk premium.

We denote by  $y^{(t)} = (y_1, \dots, y_t) \in Y$  the vector of observed up to day  $t$  (used in estimation in day  $t$ ) daily excess returns and by  $y_f^{(t)} = (y_{t+1}, \dots, y_{t+n}) \in Y_f$  the vector of forecasted observables at time  $t$ . The following density represents the  $i$ -th sampling model ( $i = 1, 2, 3, 4, 5, 6, 7$ ) at time  $t$ :

$$p(y^{(t)}, y_f^{(t)} | \theta, \eta, M_i) \prod_{j=1}^{t+n} p(y_j | \psi_{j-1}, \theta, \eta_i, M_i), \quad i = 1, 2, 3, 4, 5, 6, 7.$$

Constructed at time  $t$  Bayesian model  $M_i$ , i.e. the joint distribution of the observables  $(y^{(t)}, y_f^{(t)})$  and the vector of parameters  $(\theta, \eta_i)$  takes the form:

$$p(y^{(t)}, y_f^{(t)}, \theta, \eta_i | M_i) = p(y^{(t)}, y_f^{(t)} | \theta, \eta_i, M_i) \cdot p(\theta, \eta_i | M_i), \quad (16)$$

and requires formulation of the prior distribution  $p(\theta, \eta_i | M_i)$ , for each specification  $M_i$ . In general we assumed the following prior independence:

$$p(\theta, \eta_i | M_i) = p(\theta | M_i) \cdot p(\eta_i | M_i), \quad i = 1, 2, 3, 4, 5, 6, 7. \quad (17)$$

In this paper we assume that for each  $i$ :

$$p(\theta | M_i) = p(\theta) = p(\alpha) p(\alpha_0) p(\alpha_1) p(\beta_1) p(\nu),$$

where  $p(\alpha)$  is normal with zero mean and variance 10,  $p(\alpha_0)$  is exponential with mean 1,  $p(\alpha_1)$  and  $p(\beta_1)$  are both uniform over  $(0, 1)$ , while  $p(\nu)$  defines exponential distribution with mean (and standard deviation) equal to 10.

In order to make posterior inference about conditional asymmetry, as well as to compare prior information about this phenomenon in all competing

specifications, we considered skewness measure proposed by Arnold and Groenvelde (1995). Such a skewness measure, applied to the density  $p(y_j|\psi_{j-1}, \theta, \eta_i, M_i)$  takes the form:

$$\gamma_M = 1 - 2 \cdot P [\text{Mod}(y_j|M_i) | \psi_{j-1}, \theta, \eta_i, M_i], \quad (18)$$

where  $P[ \cdot | \psi_{j-1}, \theta, \eta_i, M_i]$  denotes the cumulative distribution function of the conditional distribution of daily return  $y_j$  (given the whole past, parameters and model  $M_i$ ) and  $\text{Mod}(y_j|M_i)$  denotes the modal value of this distribution. The case of symmetry is defined for  $\gamma_M = 0$ , while  $\gamma_M < 0$  or  $\gamma_M > 0$  imply left or right skewness of  $p(y_j|\psi_{j-1}, \theta, \eta_i, M_i)$ .

Initially, we use (18) to control the total prior information about asymmetry, which is included in each specification through the distribution  $p(\eta_i|M_i)$ . Our goal is to introduce prior information about  $\eta_i$  in such a way, that the resulting prior distribution of skewness measure  $\gamma_M$ , elicited in all models, are as much similar as it is possible. In order to perform this task the prior distribution of model specific (skewness) parameters can be defined in the following way. For  $i = 1$ ,  $\eta_1 = \gamma_1 > 0$ , and  $p(\eta_1|M_1)$  is the density of the standardized lognormal distribution truncated to the interval  $\gamma_1 \in (0.5; 2)$ . For  $i = 2$ ,  $\eta_2 = \gamma_2 \in R$ , and  $p(\eta_2|M_2)$  is the density of the normal distribution with zero mean and variance equal to 3. For  $i = 3$ ,  $\eta_3 = \gamma_3 > 0$ , and  $p(\eta_3|M_3)$  is the density of the standardized lognormal distribution. For  $i = 4$ ,  $\eta_4 = (a, b)$ , and  $p(\eta_4|M_4)$  is the product of the densities of the standardized lognormal distribution. For  $i = 5$ ,  $\eta_5 = (w_1, w_2)$  and  $p(\eta_5|M_5)$  is the product of the normal densities, both with mean 0.33 and variance 36, truncated by the following set of restrictions:  $w_1 > 0$ ,  $w_2 > 0$ ,  $w_1 + w_2 < 1$ . For  $i = 6$ ,  $\eta_6 = \gamma_4 \in R$ , and  $p(\eta_6|M_6)$  is the density of the normal distribution with mean 0 and variance equal to 20. For  $i = 7$ ,  $\eta_7 = \gamma_5 \in (-1, 1)$ , and  $p(\eta_7|M_7)$  is the density of the uniform distribution truncated to  $\gamma_5 \in [-0.5, 0.5]$ . All competing specifications, together with the prior distributions for  $\eta_i$  are presented in Table 1.

#### 4. EMPIRICAL RESULTS FOR WIG DATA

In this part we present an empirical example of Bayesian comparison of all competing specifications. We also discuss the results of the total impact of the conditional skewness assumption on the relationship between risk and return on the Warsaw Stock Exchange (WSE). Our dataset was constructed on the basis of  $T = 2144$  observations of daily growth rates,  $r_j$ , of the index of the WSE (WIG index) from 06.01.98 till 31.07.06. The risk free interest rate,  $r_j^f$ , used in excess return  $y_j$ , was approximated by the WIBOR overnight interest rate (WIBORo/n instrument). The time series of the excess returns  $y_j$ , as well as the risk free interest rate, are depicted on Figure 1. As seen from the grey plot in Figure 1,

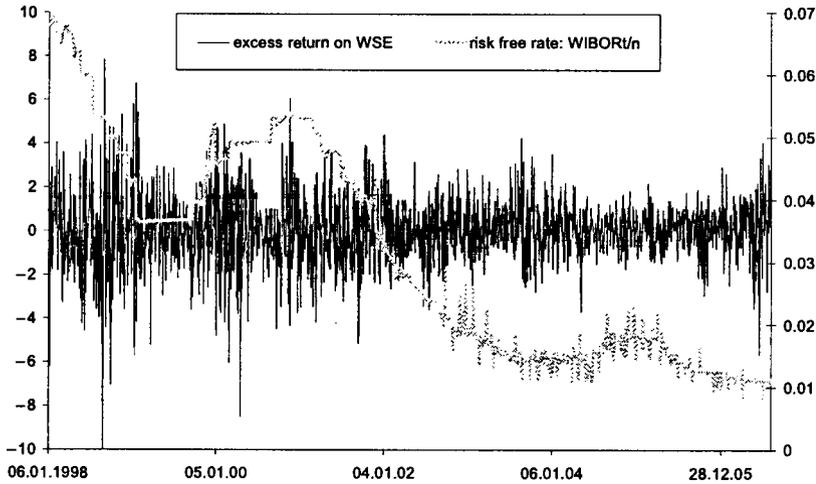


Fig. 1. Daily excess return on the Warsaw Stock Exchange from 06.01.1998 to 31.07.2006;  $T = 2144$  observations (black plot) and annualised risk free interest rate approximated by WIBOR short term interest rate (grey plot)  
 mean = 0.0214; std. dev. = 1.49; skewness = -0.28; kurtosis = 6.58

huge outliers, caused by changes in the monetary policy, together with the regions of almost no variability at the end of the 90s, depict very volatile behaviour of the Polish zloty short term interest rate. Thus, it was very important to check the sensitivity of our results with respect to the changes in the definition of the risk free interest rate. Our empirical results remained practically unchanged for  $r_f^j$  calculated on the basis of the middle and long term WIBOR zloty interest rate and also in the case  $r_f^j = 0$  for each  $j$ .

Table 2 presents decimal logarithms of the marginal data density value, as well as the posterior probabilities  $P(M_i|y^{(t)})$ , both calculated for each of competing models  $M_i$ ,  $i = 0, 1, \dots, 7$ . The initial specification  $M_0$ , built on the basis of the conditional symmetric student- $t$  distribution, receives a little data support, as the posterior probability  $P(M_0|y^{(t)})$  is not greater than 8.5%. The all remaining posterior probability mass is attached to specifications which allow for conditional skewness. It is clear, that the modelled dataset of excess returns of WIG index do not support decisively superiority of any of competing skewing mechanism. The mass of posterior probabilities is rather dispersed and strongly distributed among models which allow for conditional asymmetry. The greatest value of  $P(M_i|y^{(t)})$  receives conditionally skewed Student- $t$  GARCH model generated by the Beta distribution transformation with two free parameters. In this case the value of posterior probability is greater than 40%. The dataset also support conditionally skewed Student- $t$  GARCH model with hidden truncation mechanism ( $M_2$ ) and Beta distribution transformation with one free parameter

Table 1

Presentation of the set of competing skewing mechanisms and the prior distributions of skewness parameters,  $p(\eta_i|M_i)$ ,  $i = 1, 2, 3, 4, 5, 6, 7$

<p><math>M_2</math> Azzalini (1985), <math>\gamma_2 \in R</math>, <math>\gamma \sim N(0,3)</math></p> <p><math>p(y \gamma_2) = 2 \cdot F(\gamma_2) \cdot F^{-1}(y) \theta</math></p> <p>symmetry: <math>\gamma_2 = 0</math></p>	<p><math>M_3</math> Beta distribution with one parameter (Jones 2004), <math>\gamma_3 &gt; 0</math> <math>\ln \gamma_3 \sim N(0,1)</math></p> <p><math>p(y \gamma_3) = Be(y \gamma_3, \gamma_3^{-1})</math></p> <p>symmetry: <math>\gamma_3 = 1</math></p>	<p><math>M_6</math> Ferreira and Steel (2006) construct, <math>\gamma_4 \sim N(0,20)</math></p> <p><math>p(y \gamma_4) = 1+l(\gamma_4)[g(y \gamma_4) - 1]</math></p> <p>symmetry: <math>\gamma_4 = 0</math></p>	<p><math>M_5</math> Bernstein densities (2 parameters) <math>w_1 \sim N(0.33;36)</math>, <math>w_2 \sim N(0.33;36)</math></p> <p><math>p(y w_1, w_2) = w_1 Be(y 1,3) + w_2 Be(y 2,2) + (1-w_1-w_2)Be(y 3,1)</math></p> <p>symmetry: <math>w_i = 1/3</math>, <math>i = 1,2,3</math></p>
<p><math>M_4</math> Beta distribution, two parameters (Jones, Faddy 2003) <math>\ln a \sim N(0,1)</math>, <math>\ln b \sim N(0,1)</math>, <math>a &gt; b</math>, <math>b &gt; 0</math></p> <p><math>p(y a,b) = Be(y a,b)</math></p> <p>symmetry: <math>a = b = 1</math></p>	<p><math>M_7</math> Hansen (1994) skewed Student-t <math>\gamma_5 \sim U(-0.5; 0.5)</math></p> <p><math>p(y \gamma_5) = \frac{f\left(\frac{F^{-1}(y)}{1-\gamma_5}\right)I_{(0;0.5)}(y) + f\left(\frac{F^{-1}(y)}{1+\gamma_5}\right)I_{(0.5;1)}(y)}{f(F^{-1}(y))}</math></p> <p>symmetry: <math>\gamma_5 = 0</math></p>	<p><math>M_1</math> Inverse scale factors, Fernández and Steel (1998), <math>\ln \gamma_1 \sim N(0,1)</math>, <math>\gamma_1 \in (0.5;2)</math></p> <p><math>p(y \gamma_1) = \frac{2}{\gamma_1 + \gamma_1^{-1}} \frac{\{f\gamma_1 \cdot F^{-1}(y)I_{(0;0.5)} + f\gamma_1^{-1} \cdot F^{-1}(y)I_{(0.5;1)}\}}{f(F^{-1}(y))}</math></p> <p>symmetry: <math>\gamma_1 = 1</math></p>	

( $M_3$ ). Those three models cumulate more than 90% of the posterior probability mass, making all remained conditionally skewed specifications improbable in the view of the data. Thus, inverse scale factors (models  $M_1$  and  $M_7$ ), the Bernstein density transformation (with 2 free parameters) and Ferreira and Steel (2006) construct lead to very doubtful explanatory power of the resulting GARCH specification. Those models are strongly rejected by the data, as the values of posterior probabilities are much smaller than posterior probability of symmetric GARCH model ( $M_0$ ).

In Table 3 we present the results of Bayesian inference about tails and skewness of the conditional distribution of daily excess returns in all competing specifications. Apart from making inference about model specific skewness parameters in each model, we also put posterior means and standard deviations of Arnold and Groenvelde (1995) skewness measure  $\gamma_M$ , as well as the values of posterior probability of left asymmetry of the density  $p(y_j|\psi_{j-1}, \theta, \eta_i, M_i)$  (i.e.  $P(\gamma_M < 0|y^{(j)}, M_i)$ ).

In case of conditional symmetry (model  $M_0$ ) the dataset clearly support the hypothesis of existence of the variance of the distribution  $p(y_j|\psi_{j-1}, \theta, M_0)$ , because the whole density of the posterior distribution of the degrees of freedom parameter  $\nu$  is located on the right side of the value  $\nu = 2$ . Also, rather tight location of  $p(\nu|y^{(j)}, M_0)$  around the value  $\nu = 7$  in model  $M_0$ , assures that the conditional distribution of daily returns possesses moments of order till 7. Those properties of the posterior distribution  $p(\nu|y^{(j)}, M_0)$  remains unchanged in case of all conditionally skewed specifications. Only in model  $M_4$ , Beta distribution transformation with 2 free parameters both, location and scale of the posterior density of the degrees of freedom parameter change substantially. However the moment structure of the conditional distribution if  $y_j$  remains rather the same in model  $M_4$ . The dispersion of the posterior distribution of  $\nu$ , as measured by the posterior standard deviation, precludes conditional normality in all competing specifications.

The posterior means and standard deviations of both, asymmetry parameters  $\eta_i$  and skewness measure  $\gamma_M$  (see Arnold and Groenvelde 1995) indicate, that, in the majority of the specifications, there is strong evidence in favour of left skewness of the conditional distribution of modelled daily returns. The posterior distributions of  $\gamma_M$  are tightly located on the left side of the value  $\gamma_M = 0$  in case of  $M_i$ , for  $i = 2, 3, 4, 5$  and  $6$ , decisively confirming left asymmetry of  $p(y_j|\psi_{j-1}, \theta, \eta_i, M_i)$ . The greatest intensification of conditional skewness, measured by posterior mean of  $p(\gamma_M|y^{(j)}, M_i)$ , is obtained in model  $M_3$ . In this case the posterior expectation of asymmetry measure is equal to  $\gamma_M = -0.0216$ , with posterior standard deviation equal about 0.0024. Also hidden truncation mechanism and Beta distribution transformation with two free parameters support comparable level of conditional left asymmetry. The inverse scale factors mechanisms (both,  $M_1$  and  $M_7$ ) and conditional density based on the Ferreira and Steel (2006) construct generated posterior distributions of  $\gamma_M$ , localized very close to the value  $\gamma_M = 0$  and also much more dispersed. Consequently, in case of models  $M_1, M_6$

Table 2

Decimal logarithm of the marginal data density values, posterior probabilities of all competing specifications and posterior odds ratios for  $M_i$ , for  $i = 1, 2, 3, 4, 5, 6, 7$ , against  $M_0$

	$M_2$ Azzalini (1985)	$M_3$ Beta with 1 parameter, Jones (2004)	$M_4$ Beta with 2 parameters Jones and Faddy (2003)	$M_5$ Bernstein densites	$M_1$ Fernández and Steel (1998)	$M_6$ Ferreira and Steel (2006)	$M_7$ Hansen (1994)	$M_0$ Student- $t$ GARCH (conditional symmetry)
Symmetry	$\gamma_2 = 0$	$\gamma_3 = 1$	$a = b = 1$	$w_i = 1/3$	$\gamma_1 = 1$	$\gamma_4 = 0$	$\gamma_5 = 0$	Always
$\log p(y^{(t)} M_i)$	-1558.50	-1558.78	-1558.41	-1560.82	-1559.45	-1560.45	1559.34	-1559.06
$P(M_i y^{(t)})$ , $i = 0, \dots, 6$	0.3015	0.1582	0.3709	0.0014	0.0338	0.0338	0.0436	0.0830
$P(M_i y^{(t)})$ , $i = 1, \dots, 6$	0.3288	0.1725	0.4045	0.0016	0.0369	0.0369	0.0475	X

Table 3

Posterior means and standard deviation of tails and asymmetry parameters and posterior probabilities of left asymmetry of the density  $p(y_j | \psi_{j-1}, \theta, \eta_i, M_i)$

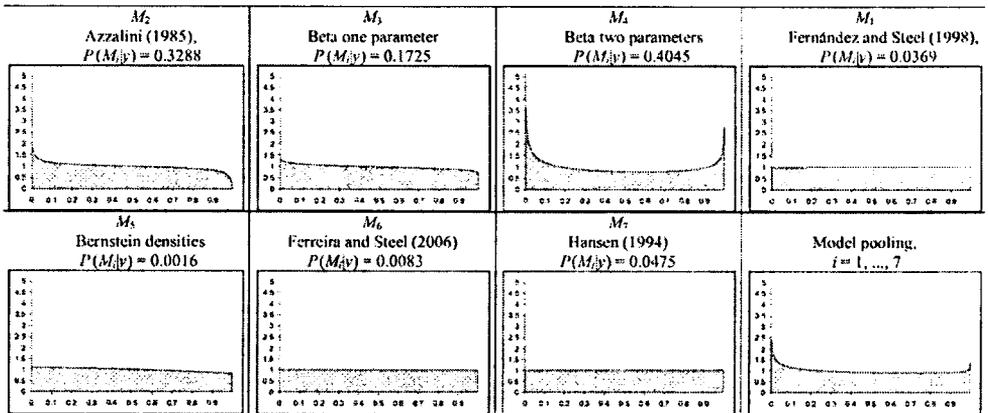
	$M_2$ Azzalini (1985)	$M_3$ Beta with 1 parameter, Jones (2004)	$M_4$ Beta with 2 parameters Jones and Faddy (2003)	$M_5$ Bernstein densites	$M_1$ Fernández and Steel (1998)	$M_6$ Ferreira and Steel (2006)	$M_7$ Hansen (1994)	$M_0$ Student- <i>t</i> GARCH (conditional symmetry)
$P(M_i   y^{(t)})$	0.3015	0.1582	0.3709	0.0014	0.0338	0.0076	0.0436	0.0830
tails $\nu > 1$	7.05 1.06	7.14 1.06	10.57 2.59	7.05 0.90	6.94 1.01	7.15 1.02	6.94 1.05	6.96 1.02
skewness, $\eta_i$	g2: -0.1074 0.0617	g3: 0.9480 0.0288	$a$ : 0.6665 0.1259 $b$ : 0.7586 0.1384	$w_1$ : 0.3637 0.0324 $w_2$ : 0.3579 0.0342	$\gamma_1$ : 1.0029 0.0148	$\gamma_4$ : -0.0489 0.0238	g5: 0.0030 0.0148	—
$\gamma_M$ , symetry $\gamma_M = 0$	-0.0208 0.0029	-0.0216 0.0024	-0.0193 0.0067	-0.0164 0.0054	0.0038 0.0080	8.086e-5	0.0030 0.0148	—
$P(\gamma_M < 0   y^{(t)}, M_i)$	1.0000	1.0000	0.9960		0.3260	14.557e-5	0.4173	—
$P(\gamma_M < 0   y^{(t)})$	0.9677							

and  $M_7$ , the posterior probabilities of left asymmetry,  $P(\gamma_M < 0 | y^{(0)}, M_i)$ , are very small, making symmetry, as well as skewness to the right, not strongly rejected by the data. Finally, on the basis of the Bayesian model pooling technique, we obtained posterior probability of left asymmetry calculated considering the whole class of specifications  $M_i, i = 1, \dots, 7$ . The modelled dataset clearly supports left asymmetry, as  $P(\gamma_M < 0 | y^{(0)}) = 0.9677$ , but it also leaves some uncertainty about the true intensification of this phenomenon. Posterior probability of symmetry and right skewness of  $p(y_j | \psi_{j-1}, \theta, \eta_i, M_i)$  (equal to 0.0323) does not totally reject those cases.

Our inverse probability integral approach to representation of the univariate skewness enables to present the sources of possible conditional asymmetry easily. In Table 4 we discuss the empirical differences between all competing skewing mechanisms. We present the posterior means of the skewing distributions  $p(.|\eta_i)$  in all competing specifications  $M_i, i = 1, \dots, 7$ . The WIG excess returns data contain specific information about the conditional asymmetry occurred as a source of different tail behavior of the density  $p(y_j | \psi_{j-1}, \theta, \eta_i, M_i)$ . This may explain such clear separation of conditionally skewed models with reasonable data support and models strongly rejected in the view of the data. The subset of specifications with very weak data support, namely models  $M_1, M_5, M_6$  and  $M_7$ , show no conditional skewness effect, because the presented densities  $p(.|\eta_i)$  are very close to the one corresponding to uniform distribution. Consequently, two alternative inverse scale factors, Ferreira and Steel (2006) construct and Bernstein densities do not yield a mechanism sensitive to the skewness represented by the WIG data. Three models with the greatest data support (built on the basis of hidden truncation mechanism and Beta distribution transforma-

Table 4

Posterior means of skewing mechanisms  $p(.|\eta_i)$  in models  $M_i, i = 1, 2, 3, 4, 5, 7$  and the skewing mechanism obtained using Bayesian model pooling approach



tions) show substantial difference of  $p(.|\eta_i)$  from the case of conditional symmetry. Since the values of  $p(.|\eta_i)$  (for  $i = 2, 3$  and  $4$ ) exhibit variability on the bounds of the interval  $(0, 1)$ , the considerable amount of skewness is located in the tails of the conditional distribution of the excess returns  $y_j$ . Quite similar tail behavior can be observed in case of hidden truncation mechanism and Beta distribution transformation with one free parameter. In model  $M_2$  and  $M_3$  the conditional left asymmetry of the density  $p(y_j|\psi_{j-1}, \theta, \eta_i, M_i)$  is forced by greater concentration of probability mass in its left tail than in the right tail. In model with the greatest data support, namely in  $M_4$  based on the Beta distribution transformation with two free parameters, the conditional skewness effect is also the result of asymmetric tail monotonicity. However the distinction between left and right tail of  $p(y_j|\psi_{j-1}, \theta, \eta_4, M_4)$  is definitely more subtle. In case of model  $M_4$ , the skewing mechanism  $p(.|\eta_4)$  makes conditional distribution of excess returns more leptokurtic, as the function  $p(.|\eta_4)$  has its extremes on the bounds of the interval  $(0, 1)$ . Since the global extreme value of  $p(y|\eta_4)$  is reached for  $y = 0$ , the skewing mechanism in  $M_4$  forces left asymmetry.

Finally in Table 5 we compare the total impact of the conditional skewness effect on the tested relation between risk and return. According to our assumptions, the conditional expectation of the excess return is proportional to the square root of the inverse precision  $h_j$ . Since we parameterize the market risk by a more general dispersion measure, than standard deviation, we report the information about the relative risk aversion by the posterior characteristics of the function  $\alpha + E(z_j)$ , see (15). Initially we checked the strength of the relation in model  $M_0$ , which does not allow for conditional skewness. Given  $M_0$ ,  $E(z_j) = 0$  and the whole information about relative risk aversion is reflected in parameter  $\alpha$ , see (13). Just like many other researchers, given  $M_0$ , we obtain positive but rather weak relation between expected excess return and risk. The posterior probability  $P(\alpha > 0|M_0, y^{(n)})$  equal about 0.92 leaves considerable level of uncertainty about the true strength of tested relation. Consequently, model  $M_0$  does not confirm our hypothesis strongly. Imposing unreasonable (in the view of the data) skewness into conditional distribution of excess returns also may not strengthen our inference. In case of models with weak data support (inverse scale factors  $M_1$  and  $M_7$ , Ferreira and Steel (2006) construct  $M_6$ ) the assumption of asymmetry of the density  $p(y_j|\psi_{j-1}, \theta, \eta_i, M_i)$  does not improve posterior inference about the sign of  $\alpha + E(z_j)$ . In case of  $M_1$ ,  $M_6$  and  $M_7$  posterior probability of positive relationship is very close to the value generated within  $M_0$ . Only in case of the skewing mechanisms with the greatest data support, namely Beta transformation with two parameters and hidden truncation, the WIG excess returns yield decisive support of the positive sign of the relative risk aversion coefficient. In case of model  $M_3$ , the posterior probability of positive sign of  $\alpha + E(z_j)$  is greater than 0.99, leaving no doubt about the significance of the relationship between risk and return postulated by Merton (1973). Hence,

Posterior analysis of the impact of the conditional skewness assumption on the relation between risk and return

	$M_2$ Azzalini (1985)	$M_3$ Beta with 1 parameter, Jones (2004)	$M_4$ Beta with 2 parameters Jones and Faddy (2003)	$M_5$ Bernstein densites	$M_1$ Fernández and Steel (1998)	$M_6$ Ferreira and Steel (2006)	$M_7$ Hansen (1994)	$M_0$ Student-t GARCH (conditional symmetry)
$\alpha + E(z_t)$	0.2567 0.1162	0.1440 0.0912	0.2148 0.0852	0.2085 0.0933	0.0469 0.0349	0.0489 0.0338	0.441 0.0316	0.0483 0.0337
$P(\alpha + E(z_t) > 0   M_i, y^{(t)})$	0.9894	0.9528	0.9972	0.9893	0.9102	0.9230	0.9171	0.9201
$P(\alpha + E(z_t) > 0   y^{(t)})$	0.9777							

it is possible to confirm positive sing of  $\alpha + E(z_i)$  only by imposing specific skewing mechanism into conditional distribution of excess returns. Beta distribution transformation with two free parameters is able to detect additional source of information about risk premium in the WIG dataset. Also, hidden truncation mechanism and Bernstein density transformation strongly confirm positive sing of the risk aversion coefficient, as posterior probability  $P(\alpha + E(z_i) > 0 | M_i, \gamma^{(i)})$  is greater than 0.98 for  $i = 2$  and 5.

## 5. CONCLUDING REMARKS

In this paper we presented the results of Bayesian estimation of the impact of the conditional skewness assumption on the strength of the relationship between risk and return. Initially, on the basis of the Intertemporal CAPM model, proposed by Merton (1973), we built the GARCH-In-Mean sampling model, suitable in analysing such relationship. Our approach, which fully relates to the model proposed by Osiewalski and Pipień (2000), treats the skewness of the conditional distribution of excess returns as an alternative source of information about risk aversion. Thus pure statistical feature of the sampling model was equipped with economic interpretation.

Based on the daily excess returns of WIG index we checked the total impact of conditional skewness assumption on the relation between risk and return. Posterior inference about skewing mechanisms showed positive and decisively significant value of the coefficient of the relative risk aversion once an appropriate, specific skewing mechanism was imposed in conditional Student- $t$  distribution. The greatest data support, and also very strong support of the relation postulated by Merton (1973), received skewness obtained by Beta distribution transformation with two free parameters; see Jones and Faddy (2003). Also strong confirmation of positive relative risk aversion coefficient was generated by hidden truncation mechanism, proposed by Azzalini (1985) and Bernstein density transformation. Many other skewing mechanisms considered in the paper were strongly rejected by the WIG dataset. Consequently, in many cases we observed no impact of conditional asymmetry on the tested relation. The WIG excess return contained specific information about conditional skewness, which was able to detect only by models with the greatest data support. Thus, in our empirical example it was possible to identify asymmetry of the conditional distribution as an additional source of information concerning risk premium only in case of specific, more flexible skewing mechanisms, than previously considered in the literature.

## STRESZCZENIE

Mateusz Pipień. *Wykorzystanie warunkowej asymetrii w badaniu zależności pomiędzy oczekiwaną stopą zwrotu a poziomem ryzyka. Analiza bayesowska dla danych WIG*. Folia Oeconomica Cracoviensia 2007, 48: 95–117.

Zasadniczym celem artykułu jest zastosowanie wnioskowania bayesowskiego w badaniu relacji pomiędzy oczekiwaną stopą zwrotu a poziomem ryzyka na Gieldzie Papierów Wartościowych w Warszawie. Na podstawie międzyokresowego modelu CAPM (ang. *Intertemporal-CAPM*), zaproponowanego przez Mertona (1973), zbudowano ogólny model próbkowy w ramach którego jest możliwe testowanie omawianej relacji. W rozważanym modelu statystycznym źródłem zależności pomiędzy oczekiwanym zwrotem a poziomem ryzyka jest asymetria rozkładu warunkowego stóp zmian. Propozycja ta wykorzystuje koncepcję GARCH-In-Mean Osiewalskiego i Pipienia (2000) oraz dostarcza formalnej, ekonomicznej interpretacji skośności występującej w modelu próbkowym.

W artykule rozważono wiele konkurencyjnych specyfikacji dopuszczających warunkową asymetrię. W szczególności zastosowano mechanizm ukrytego ucięcia, alternatywne skalowania wokół modalnej, transformacje rozkładem Beta, gęstości Bersteina oraz metodę zaproponowaną przez Ferreira i Steela (2006).

Na podstawie danych dotyczących kwotowań indeksu WIG dokonano analizy wpływu wprowadzenia do modelu warunkowej asymetrii stóp zmian na postulowaną relację pomiędzy oczekiwaną stopą zmian a poziomem ryzyka. Wyniki bayesowskiego porównania modeli, jak również analiza *a posteriori* parametrów skośności potwierdziły w sposób zdecydowany założenia teorii Mertona (1973). Uzyskano silnie istotną i dodatnią zależność pomiędzy stopą zwrotu indeksu WIG i zmiennością. Najwyższą wartość prawdopodobieństwa *a posteriori* uzyskał model, w którym efekt asymetrii jest rezultatem zastosowania transformacji rozkładem Beta z dwoma swobodnymi parametrami.

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